

Alternative ion acoustic solitary waves behaviour in presences of Landau damping

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Abstract

Bandyopadhyay and Das [Phys. Plasmas, **9**, 465-473, 2002] have derived a nonlinear macroscopic evolution equation for ion acoustic wave in a magnetized plasma consisting of warm adiabatic ions and non-thermal electrons including the effect of Landau damping. In that paper they have also derived the corresponding nonlinear evolution equation when coefficient of the nonlinear term of the above mentioned macroscopic evolution equation vanishes, the nonlinear behaviour of the ion acoustic wave is described by a modified macroscopic evolution equation. But they have not considered the case when the coefficient is very near to zero. This is the case we consider in this paper and we derive the corresponding evolution equation including the effect of Landau damping. Finally, a solitary wave solution of this macroscopic evolution is obtained, whose amplitude is found to decay slowly with time. The corresponding nonlinear evolution equation and its solitary wave solution without including the effect of Landau damping has been subject matter of study of the paper made by Das *et al.* [Phys. Plasmas, **14**, 092304-1--10, 2007]

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1. Introduction

In the investigations made by Das *et al* [2], the Landau damping effect has not been taken into account. In the present paper, we include this effect on the problem considered in Das *et al* [2]. Starting from the same governing equations but replacing the expression for the number density of non-thermal electrons by the Vlasov-Boltzmann equation for electrons, an appropriate macroscopic evolution equation corresponding to the combined modified Korteweg-de Vries-Zakharov-Kuznetsov (combined MKdV-KdV-ZK) equation of Das *et al* [2] is derived, which describes the long-time evolution of weakly nonlinear long wave-length ion acoustic waves in a magnetized plasma consisting of warm adiabatic ions and non-thermal electrons including the effect of Landau damping. Following Ott and Sudan [3], Bandyopadhyay and Das [1] derived a macroscopic evolution equation to investigate the nonlinear behaviour of the ion acoustic waves in a magnetized plasma consisting of warm adiabatic ions and non-thermal electrons including the effect of Landau damping. This equation is a Korteweg-de Vries-Zakharov-Kuznetsov (KdV-ZK) equation except for an extra term that accounts for the effect of Landau damping. Bandyopadhyay and Das [1] reported that this macroscopic evolution equation admits solitary wave solution propagating obliquely to the external uniform static magnetic field and having a sech^2 -profile. But the amplitude of solitary wave does not remain constant; it varies slowly with time τ as $(1 + \tau/\tau_0)^{-2}$, where τ_0 is a constant depending on the initial amplitude of the solitary wave, the angle between the direction of propagation of the solitary wave and the

external uniform static magnetic field and the parameters involved in the system. This evolution equation, which we are discussing about, loses its validity when the coefficient of the nonlinear term of the macroscopic evolution equation vanishes and this vanishes along a particular curve in the $\beta\sigma$ -parametric plane as shown in Fig1. In this situation, in the same paper, they have derived a modified macroscopic evolution equation when the coefficient of the nonlinear term of the macroscopic evolution equation vanishes. This equation is a modified Korteweg-de Vries-Zakharov-Kuznetsov (MKdV-ZK) equation except for an extra term that accounts for the effect of Landau damping. Bandyopadhyay and Das [2] reported that this modified macroscopic evolution equation admits solitary wave solution propagating obliquely to the external uniform static magnetic field and having a *sech*-profile. But the amplitude of solitary wave does not remain constant; it varies slowly with time τ as $(1 + \tau/\tau_0)^{-1}$, where τ_0 is a constant depending on the initial amplitude of the solitary wave, the angle between the direction of propagation of the solitary wave and the external uniform static magnetic field and the parameters involved in the system. But again this modified macroscopic evolution equation is unable to describe the nonlinear behaviour of the ion acoustic waves including the effect of Landau damping if the coefficient of the nonlinear term of the macroscopic evolution equation approaches to zero, but not exactly equal to zero. This is the case we are considering in the present paper; we derive a further modified macroscopic evolution equation to study the nonlinear behaviour of the ion acoustic waves including the effect of Landau damping. This equation is a combined MKdV-KdV-ZK equation except for an extra term that accounts for the effect of Landau damping. The solitary wave solution of this further modified macroscopic evolution has been obtained. It is found that due to inclusion of the effect of Landau damping the amplitude of the alternative solitary wave solution of this equation is a slowly varying function of time.

The physics of nonlinear Landau damping is of interest for two major reasons. First, it is a fundamental and distinctive plasma phenomenon that links collective and single-particle behaviour. Second, the derivation of reduced fluid models that incorporate accurately such kinetic effects, is of great importance for plasma transport studies. For instance, some authors have proposed a k -dependent dissipation term, which correctly reproduces linear Landau damping within the framework of fluid models [4]. However, the long time behaviour of Landau damping is intrinsically nonlinear, and, in order to assess the validity of the above models, it is important to understand whether the damping will continue indefinitely, or will eventually be stopped by the nonlinearity. In the nonlinear theory, it is well known that the usual ion acoustic wave in electron ion plasma suffers Landau damping due to the resonant particle i.e. the particles whose velocity is nearly equal to the wave phase velocity (Chen[11]). Ott and Sudan [3] have incorporated this linear Landau damping on ion acoustic solitary wave neglecting the particle trapping effect under the assumption that the particle trapping time is much greater than the Landau damping time.

The research works on solitary waves in plasmas have been done under various physical conditions such as plasmas including multi-species ions [5], negative ions [6], and dust particles [7]. In many cases, the Korteweg-de Vries (KdV) equation is used to describe basic characters of the wave. Detailed properties of the solitary waves observed in experiments in plasmas are, however, slightly different from those predicted by the equation. Using Q-machine plasmas, Karpman *et al* [8] have observed oscillations in the tail of solitary waves, which are caused by resonant particles and have shown that the tail changes its shape depending on the strength of Landau damping.

This paper is an extension of the work of Bandyopadhyay and Das [2], where we derive a further modified macroscopic evolution equation which describe the non-linear behaviour of ion-acoustic waves in fully ionized collisionless plasma consisting of warm adiabatic ions and non-thermal electrons having vortex-like velocity distribution, immersed in a uniform static magnetic field directed along z -axis including the effect of Landau damping. This equation is true only for the case when the coefficient of the nonlinear term of the macroscopic evolution equation derived by Bandyopadhyay and Das [2] approaches to zero and not exactly equal to zero. With the help of multiple time scale analysis of Ott and Sudan [3], we find a solitary wave solution of this equation. From the solution, we can conclude that the amplitude of the solitary wave slowly decreases with time.

This paper is organized as follows. The basic equation have been given in Section 2. The macroscopic evolution equations are given in Section 3, in which the derivation of MKdV-KdV-ZK like macroscopic evolution equation is given in subsection 3.3. Solitary wave solutions of the combined MKdV-KdV-ZK like macroscopic evolution equation are investigated in Section 4. Finally, we have concluded our findings in Section 5.

2. Basic Equation

The following are the governing equations describing the non-linear behaviour of ion-acoustic waves in fully ionized collisionless plasma consisting of warm adiabatic ions and non-thermal electrons having vortex-like velocity distribution, immersed in a uniform static magnetic field directed along z -axis. Here it is assumed that the plasma beta i.e., the ratio of particle pressure to the magnetic pressure is very

small and the characteristic frequency is much smaller than ion cyclotron frequency (Cairns *et al* [9], Mamun [10])

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\varphi + \omega_c(\mathbf{u} \times \hat{z}) - \frac{\sigma}{n}\nabla p, \tag{2}$$

$$\nabla^2\varphi = n_s - n, \tag{3}$$

$$p = n^\gamma, \tag{4}$$

where

$$n_s = \int_{-\infty}^{\infty} f dv_{\parallel} \tag{5}$$

and the velocity distribution function of electrons f must satisfy the Vlasov- Boltzmann equation.

$$\sqrt{\frac{m_e}{m_i}} \frac{\partial f}{\partial t} + v_{\parallel} \frac{\partial f}{\partial z} + \frac{\partial \varphi}{\partial z} \frac{\partial f}{\partial v_{\parallel}} = 0. \tag{6}$$

Here $n, n_s, \mathbf{u}, p, \varphi, (x, y, z)$ and t are respectively the ion number density, electron number density, ion fluid velocity, ion pressure, electrostatic potential, spatial variables and time, and they have been normalized respectively by n_0 (unperturbed ion number density), $n_0, c_s (= \sqrt{\frac{K_B T_e}{m}})$ (ion-acoustic speed), $n_0 K_B T_i, \frac{K_B T_e}{e}, \lambda_D (= \sqrt{\frac{K_B T_e}{4\pi n_0 e^2}})$ (Debye length) and ω_p^{-1} (ion plasma period), where $\sigma = \frac{T_i}{T_e}, \omega_c$ is the ion cyclotron frequency normalized by $\omega_p (= \sqrt{\frac{4\pi n_0 e^2}{m}})$ and $\gamma (= \frac{5}{3})$ is the ratio of two specific heats. Here K_B is the Boltzmann constant; T_e, T_i are respectively the electron and ion temperatures; m is the mass of an ion and e is the electronic charge v_{\parallel} is the velocity of electrons in phase space normalized to $v_s = \sqrt{\frac{K_B T_e}{m_e}}$. In (4), the adiabatic law has been taken on the basis of the assumption that the effect of viscosity, thermal conductivity and the energy transfer due to collision can be neglected.

Since the electrons are assumed to be nonthermally distributed, the electron velocity distribution function can be taken as (Cairns *et al* [9])

$$f_0(v_{\parallel}) = \frac{1}{\sqrt{2\pi}(1 + 3\alpha_1)} (1 + \alpha_1 v_{\parallel}^4) e^{-\frac{1}{2}v_{\parallel}^2}, \tag{7}$$

where $\alpha_1 (\geq 0)$ is a parameter that determines proportion of fast energetic nonthermal electrons.

To discuss the effect of Landau damping on ion-acoustic solitary waves, we follow the method of Ott and

Sudan [3] and following them, we replace $\sqrt{\frac{m_e}{m_i}}$ by $\sqrt{\frac{m_e}{m_i}} \epsilon$ where ϵ is a small parameter. The equation (6)

then assumes the following form

$$\alpha_2 \epsilon \frac{\partial f}{\partial t} + v_{\parallel} \frac{\partial f}{\partial Z} + \frac{\partial \varphi}{\partial Z} \frac{\partial f}{\partial v_{\parallel}} = 0 \tag{8}$$

where $\alpha_2 = \sqrt{\frac{m_e}{m_i}}$

Again using (4), the equation (2) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\varphi + \omega_c(\mathbf{u} \times \hat{z}) - \frac{5}{3} \sigma n^{-\frac{1}{3}} \nabla n \tag{9}$$

and the equation (3), which is the Poisson equation becomes

$$\epsilon \nabla_{\xi}^2 \varphi + n - \int_{-\infty}^{\infty} f dv_{\parallel} = 0. \tag{10}$$

Therefore Eqs. (1), (9), (10) and (8) are our governing equations.

3. Evolution equations

3.1. Macroscopic evolution equation

Before deriving the nonlinear evolution equation for ion-acoustic wave in a magnetized collisionless plasma consisting of warm adiabatic ions and non-thermal electrons including the effect of Landau damping for a particular case not considered in the paper of Bandyopadhyay and Das [1] we give below in short a summary of the results obtained in that paper. The macroscopic evolution equation obtained is the following:

$$\frac{\partial \varphi^{(1)}}{\partial \tau} + AB' \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial \zeta} + \frac{1}{2} A \frac{\partial^3 \varphi^{(1)}}{\partial \zeta^3} + \frac{1}{2} AD \frac{\partial}{\partial \zeta} \left(\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} \right) + \frac{1}{2} AE \alpha_2 P \int_{-\infty}^{\infty} \frac{\partial \varphi^{(1)}}{\partial \zeta'} \frac{d\zeta'}{\zeta - \zeta'} = 0, \quad (11)$$

where

$$A = (V(1 - \beta)^2)^{-1}, \quad (12)$$

$$B' = \frac{1}{2} \left[(1 - \beta)^3 \left(3V^2 - \frac{5\sigma}{9} \right) - 1 \right], \quad (13)$$

$$D = 1 + \frac{V^4}{\omega_c^2} (1 - \beta)^2, \quad (14)$$

$$E = \frac{V}{4\sqrt{2}\pi} (4 - 3\beta). \quad (15)$$

and the constant V is given by

$$(1 - \beta) \left(V^2 - \frac{5\sigma}{3} \right) = 1 \quad (16)$$

and

$$\beta = \frac{4\alpha_1}{1 + 3\alpha_1}$$

The equation (11) is a KdV-ZK equation except for an extra term (last term of the left hand side of (11)) that accounts for the effect of Landau damping. The solitary wave solution of the equation (11) has been obtained in that paper of Bandyopadhyay and Das [1]. They have found that the solitary wave solution of the equation (11) has the same $sech^2$ -profile as in the case of KdV-ZK equation. But, here the amplitude as well as the width of the solitary wave varies slowly with time. In particular, the amplitude (a) of the solitary wave solution of the equation (11) is given by the following equation.

$$a = a_0 \left(1 + \frac{\tau}{\tau'} \right)^{-2} \quad (17)$$

where a_0 is the value of a at $\tau = 0$ and τ' is given by the following equation

$$\tau' = \left[\frac{1}{4} \cos \delta \sqrt{\frac{A^2 E^2 \alpha_2^2 a_0 B'}{6(\cos^2 \delta + D \sin^2 \delta)} P} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} sech^2 Z \frac{\partial (sech^2 Z')}{\partial Z'} \frac{dZ' dZ}{Z - Z'} \right]^{-1} \quad (18)$$

Using (16) the expression for B' can be simplified as

$$B' = \frac{20(1 - \beta)^3}{9} (\sigma - \sigma_\beta) \quad (19)$$

where

$$\sigma_\beta = \frac{9(1 - 3(1 - \beta)^2)}{40(1 - \beta)^3} \quad (20)$$

From this expression of B' it is easy to see that the coefficient of the non-linear term in (11) vanishes along a particular curve Fig1 in the $\beta\sigma$ plane, and consequently, it is not possible to discuss the nonlinear behaviour of ion acoustic wave including the effect of Landau damping with the help of Eq. (11). In this situation, i.e., when $B' = 0$, Bandyopadhyay and Das [1] have also derived a modified macroscopic evolution equation.

3.2. Modified macroscopic evolution equation

For this case, i.e., when $B' = 0$ giving appropriate stretching of space coordinates and time, and appropriate perturbation expansions of the dependent variables Bandyopadhyay and Das [1] in the same paper have derived the following modified macroscopic evolution equation for ion acoustic waves in a fully ionized collisionless plasma consisting of warm ions and non-thermal electrons immersed in a uniform static magnetic field directed along the z-axis:

$$\frac{\partial \varphi^{(1)}}{\partial \tau} + AB'' [\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial \zeta} + \frac{1}{2} A \frac{\partial^3 \varphi^{(1)}}{\partial \zeta^3} + \frac{1}{2} AD \frac{\partial}{\partial \zeta} \left(\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} \right) + \frac{1}{2} AE \alpha_2 P \int_{-\infty}^{\infty} \frac{\partial \varphi^{(1)}}{\partial \zeta'} \frac{d\zeta'}{\zeta - \zeta'} = 0, \quad (21)$$

Here A, D and E are same as given by the equations (12), (14) and (15) respectively and B'' is given by the following equation:

$$B'' = \frac{1}{2} \left[\left(V^2 - \frac{5\sigma}{3} \right)^{-4} \left(\frac{10}{27} \sigma - 6V^2 + \frac{3}{2} \left(V^2 - \frac{5\sigma}{3} \right)^{-1} \left(3V^2 - \frac{5\sigma}{9} \right)^2 \right) - \frac{1}{2} (1 + 3\beta) \right] \quad (22)$$

and the constant V is determined from equation (16).

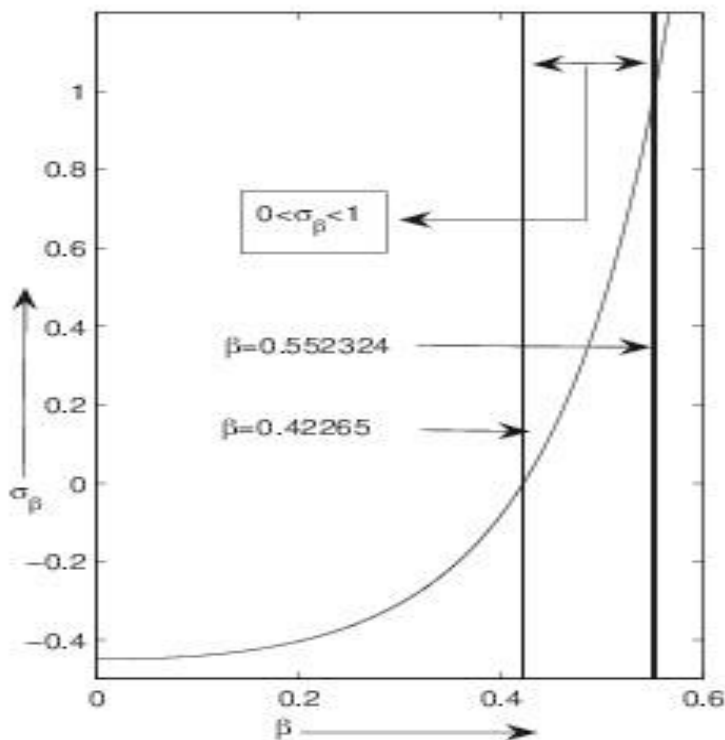


Figure 1: Variation of σ_β against β

The equation (21) is a MKdV-ZK equation except for an extra term (last term of the left hand side of (21) that accounts for the effect of Landau damping. The solitary wave solution of the equation (21) has been investigated by Bandyopadhyay and Das [1] in the same paper. They have found that the solitary wave solution of the equation (21) has the same *sech*-profile as in the case of MKdV-ZK equation. But, here the amplitude as well as the width of the solitary wave varies slowly with time. In particular, the amplitude (a) of the solitary wave solution of the equation (21) is given by the following equation.

$$a = a_0 \left(1 + \frac{\tau}{\tau'}\right)^{-1} \tag{23}$$

where a_0 is the value of a at $\tau = 0$ and τ' is given by the following equation

$$\tau' = \left[\frac{1}{2} \cos \delta \sqrt{\frac{A^2 E^2 \alpha_2^2 a_0^2 B''}{3(\cos^2 \delta + D \sin^2 \delta)}} P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{sech} Z \frac{\partial(\operatorname{sech} Z')}{\partial Z'} \frac{dZ' dZ}{Z - Z'} \right]^{-1} \tag{24}$$

But both the evolution equations (11) and (21) are unable to describe the nonlinear behaviour of the ion acoustic wave along with the effect of Landau damping in the neighbourhood of the curve $\sigma = \sigma_\beta$ in the $\beta\sigma$ -parametric plane along which $B' = 0$ (Fig1). This is the situation we are considering here. We have derive in this case, a further modified macroscopic evolution equation which describes the nonlinear behaviour of the ion acoustic wave in the neighbourhood of the curve $\sigma = \sigma_\beta$ in the $\beta\sigma$ - parametric plane along which $B' = 0$.

3.3. Further Modified Macroscopic evolution equation

To discuss the nonlinear behaviour of the ion acoustic wave in the neighbourhood of the curve in the $\beta\sigma$ parametric plane along which $B' = 0$, we assume $B' \approx O(\epsilon^{\frac{1}{2}})$ (Nejoh [11]), and we take the following stretching of space coordinates and time.

$$\xi = \epsilon^{\frac{1}{2}} x, \eta = \epsilon^{\frac{1}{2}} y, \zeta = \epsilon^{\frac{1}{2}} (z - Vt), \tau = \epsilon^{\frac{3}{2}} t \tag{25}$$

where ϵ is a small parameter measuring the weakness of the dispersion and V is a constant.

With the stretching given by (25), the equations (1), (9), (10) and (8) respectively assume the following form:

$$-\epsilon^{\frac{1}{2}} V \frac{\partial n}{\partial \zeta} + \epsilon^{\frac{3}{2}} \frac{\partial n}{\partial \tau} + \epsilon^{\frac{1}{2}} \nabla_\xi (nu) = 0, \tag{26}$$

$$-\epsilon^{\frac{1}{2}} V \frac{\partial u}{\partial \zeta} + \epsilon^{\frac{3}{2}} \frac{\partial u}{\partial \tau} + \epsilon^{\frac{1}{2}} (\mathbf{u} \cdot \nabla_{\xi}) \mathbf{u} = -\epsilon^{\frac{1}{2}} \nabla_{\xi} \varphi + \omega_c (\mathbf{u} \times \hat{z}) - \frac{5}{3} \sigma \epsilon^{\frac{1}{2}} n^{-\frac{1}{3}} \nabla n \tag{27}$$

$$\nabla_{\xi}^2 \varphi + n - \int_{-\infty}^{\infty} f dv_{\parallel} = 0 \tag{28}$$

$$-V \alpha_2 \epsilon^{\frac{3}{2}} \frac{\partial f}{\partial \zeta} + \alpha_2 \epsilon^{\frac{5}{2}} \frac{\partial f}{\partial \tau} + \epsilon^{\frac{1}{2}} v_{\parallel} \frac{\partial f}{\partial \zeta} + \epsilon^{\frac{1}{2}} \frac{\partial \varphi}{\partial \zeta} \frac{\partial f}{\partial v_{\parallel}} = 0 \tag{29}$$

Here

$$\nabla_{\xi} = \hat{e}_x \frac{\partial}{\partial \xi} + \hat{e}_y \frac{\partial}{\partial \eta} + \hat{e}_z \frac{\partial}{\partial \zeta}, \quad \mathbf{u} = (x, y, z). \tag{30}$$

Next we use the following perturbation expansions of the dependent variables to make a balance between nonlinear and dispersive terms.

$$(n, \varphi, \omega, f) = (1, 0, 0, f_0) + \sum_{i=1}^{\infty} \epsilon^i (n^{(i)}, \varphi^{(i)}, \omega^{(i)}, f^{(i)}), \quad (u, v) = \sum_{i=1}^{\infty} \epsilon^{\frac{i+2}{2}} (u^{(i)}, v^{(i)}) \tag{31}$$

Substituting (31) into the equations (26)-(29) and then equating coefficient of different powers of φ on both sides, we get a sequence of equations. From the lowest order equations obtained from (26)-(28), which are at the order ϵ , we get the following equations.

$$n^{(1)} = \left(V^2 - \frac{5\sigma}{3} \right)^{-1} \varphi^{(1)}, \tag{32a}$$

$$\omega^{(1)} = V \left(V^2 - \frac{5\sigma}{3} \right)^{-1} \varphi^{(1)}, \tag{32b}$$

$$u^{(1)} = -\frac{V^2}{\omega_c} \left(V^2 - \frac{5\sigma}{3} \right)^{-1} \frac{\partial \varphi^{(1)}}{\partial \eta}, \tag{32c}$$

$$v^{(1)} = \frac{V^2}{\omega_{\hat{e}_x}} \left(V^2 - \frac{5\sigma}{3} \right)^{-1} \frac{\partial \varphi^{(1)}}{\partial \xi}, \tag{32d}$$

$$n^{(1)} = \int_{-\infty}^{\infty} f^{(1)} dv_{\parallel} \tag{32e}$$

From the Vlasov equation (29) at the lowest order, i.e., at the order $\epsilon^{\frac{1}{2}}$, we get the following equation

$$v_{\parallel} \frac{\partial f^{(1)}}{\partial \zeta} + \frac{\partial \varphi^{(1)}}{\partial \zeta} \frac{\partial f_0}{\partial v_{\parallel}} = 0 \tag{33}$$

As this equation does not have unique solutions, we include an extra higher order term $\epsilon^3 \alpha_2 \frac{\partial f^{(1)}}{\partial \tau} = \epsilon \epsilon^2 \alpha_2 \frac{\partial f^{(1)}}{\partial \tau}$ originated from the Vlasov equation at the order ϵ^2 . Let us write the equation (33) as follows.

$$\epsilon^2 \alpha_2 \frac{\partial f_{\epsilon}^{(1)}}{\partial \tau} + v_{\parallel} \frac{\partial f_{\epsilon}^{(1)}}{\partial \zeta} + \frac{\partial \varphi^{(1)}}{\partial \zeta} \frac{\partial f_0}{\partial v_{\parallel}} = 0 \tag{34}$$

Then $f^{(1)}$ can be obtained as unique solution of this equation by imposing the natural relation of the form

$$f^{(1)} = \lim_{\epsilon \rightarrow 0} f_{\epsilon}^{(1)} \tag{35}$$

Assuming τ dependence of $f_{\epsilon}^{(1)}$ and $\varphi^{(1)}$ to be of the form $e^{i\omega\tau}$, the equation (35) can be written as

$$i\omega \epsilon^2 \alpha_2 f_{\epsilon}^{(1)} + v_{\parallel} \frac{\partial f_{\epsilon}^{(1)}}{\partial \zeta} + \frac{\partial \varphi^{(1)}}{\partial \zeta} \frac{\partial f_0}{\partial v_{\parallel}} = 0 \tag{36}$$

Now taking Fourier transform of this equation with respect to the variable ζ according to the definition,

$$\hat{g} = \hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\zeta) e^{-ik\zeta} d\zeta, \tag{37}$$

we get

$$i\omega \epsilon^2 \alpha_2 \hat{f}_{\epsilon}^{(1)} + ikv_{\parallel} \hat{f}_{\epsilon}^{(1)} + ik\hat{\varphi}^{(1)} \frac{\partial f_0}{\partial v_{\parallel}} = 0. \tag{38}$$

This equation gives the following expression for $\hat{f}_{\epsilon}^{(1)}$

$$\hat{f}_{\epsilon}^{(1)} = -\frac{k \frac{\partial f_0}{\partial v_{\parallel}}}{kv_{\parallel} + \omega \epsilon^2 \alpha_2} \hat{\varphi}^{(1)}. \tag{39}$$

Now whenever the factor $1/(kv_{\parallel} + \omega \epsilon^2 \alpha_2)$ comes under integration over v_{\parallel} along the real axis, the general prescription is to replace this integration according to Landau, along a contour in the complex v_{\parallel} -

plane known as Landau contour. This is equivalent to replacing the factor $1/(kv_{\parallel} + \omega\epsilon^2\alpha_2)$ by the following:

$$\frac{1}{kv_{\parallel} + \omega\epsilon^2\alpha_2} = P\left(\frac{1}{kv_{\parallel} + \omega\epsilon^2\alpha_2}\right) + i\pi\delta(kv_{\parallel} + \omega\epsilon^2\alpha_2) \tag{40}$$

Substituting this relation into the equation (39) and then proceeding to the limit $\epsilon \rightarrow 0+$, we get according to (35) the following expression for $\hat{f}^{(1)}$:

$$\hat{f}^{(1)} = -2\frac{\partial f_0}{\partial v_{\parallel}^2}\hat{\varphi}^{(1)} - 2\pi i\frac{\partial f_0}{\partial v_{\parallel}^2}kv_{\parallel}\delta(kv_{\parallel})\hat{\varphi}^{(1)}. \tag{41}$$

Due to the relation $x\delta(x) = 0$, this equation assumes the following form:

$$\hat{f}^{(1)} = -2\frac{\partial f_0}{\partial v_{\parallel}^2}\hat{\varphi}^{(1)} \tag{42}$$

Taking Fourier inverse transform of (42), we get

$$f^{(1)} = -2\frac{\partial f_0}{\partial v_{\parallel}^2}\varphi^{(1)} \tag{43}$$

Substituting (43) in the last equation of (32) and then performing the integration we get

$$n^{(1)} = (1 - \beta)\varphi^{(1)}, \tag{44}$$

This equation along with the first equation of (32) gives the following dispersion relation to determine the constant V

$$(1 - \beta)\left(V^2 - \frac{5\sigma}{3}\right) = 1.$$

This equation is same as the equation (16) as well as the equation (20) of Das et. al. [2].

In the next order, i.e., at the order $\epsilon^{\frac{3}{2}}$ solving the ion continuity equation and the parallel component (i.e., the component parallel to the ambient magnetic field, i.e., z-component or ζ -component) of ion fluid equation of motion for $n^{(2)}$ and $\omega^{(2)}$ to express them in terms of $\varphi^{(1)}$ and $\varphi^{(2)}$, we get the following equations:

$$n^{(2)} = \frac{1}{\left(V^2 - \frac{5\sigma}{3}\right)}\varphi^{(2)} + \frac{1}{2}\frac{\left(3V^2 - \frac{5\sigma}{9}\right)}{\left(V^2 - \frac{5\sigma}{3}\right)^3}[\varphi^{(1)}]^2, \tag{45}$$

$$\omega^{(2)} = \frac{V}{\left(V^2 - \frac{5\sigma}{3}\right)}\varphi^{(2)} + \frac{1}{2}V\frac{\left(V^2 + \frac{25\sigma}{9}\right)}{\left(V^2 - \frac{5\sigma}{3}\right)^3}[\varphi^{(1)}]^2. \tag{46}$$

From the perpendicular component (i.e., the component perpendicular to the ambient magnetic field, i.e., the components along x-axis and y-axis) of the ion fluid equation of motion at the order $\epsilon^{\frac{3}{2}}$, we get the following equation:

$$\frac{\partial u^{(2)}}{\partial \xi} + \frac{\partial v^{(2)}}{\partial \eta} = \frac{V^2}{\omega_c^2}\left(V^2 - \frac{5\sigma}{3}\right)^{-1}\frac{\partial}{\partial \zeta}\left(\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2}\right) \tag{47}$$

From the Poisson equation at the order ϵ , we get

$$n^{(2)} - \int_{-\infty}^{\infty} f^{(2)} dv_{\parallel} = 0. \tag{48}$$

To find $f^{(2)}$, we again consider the Vlasov equation at the order $\epsilon^{\frac{3}{2}}$. The Vlasov equation at the order $\epsilon^{\frac{3}{2}}$ is the following, in which as mentioned in the lowest order Vlasov equation, an extra time derivative term $\epsilon^{\frac{7}{2}}\alpha_2\frac{\partial f^{(2)}}{\partial \tau} = \epsilon^{\frac{3}{2}}\epsilon^2\alpha_2\frac{\partial f^{(2)}}{\partial \tau}$ has been included and $f^{(2)}$ has been replaced by $\hat{f}_{\epsilon}^{(2)}$

$$\epsilon^2\alpha_2\frac{\partial \hat{f}_{\epsilon}^{(2)}}{\partial \tau} + v_{\parallel}\frac{\partial \hat{f}_{\epsilon}^{(2)}}{\partial \zeta} + \frac{\partial \varphi^{(2)}}{\partial \zeta}\frac{\partial f_0}{\partial v_{\parallel}} = 2v_{\parallel}\frac{\partial^2 f_0}{\partial (v_{\parallel}^2)^2}\frac{\partial}{\partial \zeta}[\varphi^{(1)}]^2 \tag{49}$$

Then $f^{(2)}$ can be obtained as unique solution of this equation by imposing the natural relation of the form

$$f^{(2)} = \lim_{\epsilon \rightarrow 0} \hat{f}_{\epsilon}^{(2)} \tag{50}$$

Assuming τ dependence of $\hat{f}_{\epsilon}^{(2)}$ and $\varphi^{(2)}$ to be of the form $e^{i\omega\tau}$, taking Fourier transform of this equation with respect to the variable ζ , using the causality condition (40) and finally proceeding to the limit $\epsilon \rightarrow 0+$, we get according to (50) the following expression for $\hat{f}^{(2)}$:

$$\hat{f}^{(2)} = -2\frac{\partial f_0}{\partial v_{\parallel}^2}\hat{\varphi}^{(2)} + 2\frac{\partial^2 f_0}{\partial (v_{\parallel}^2)^2}\hat{a}_3 \tag{51}$$

where

$$\hat{a}_3 = [\hat{\varphi}^{(1)}]^2 \tag{52}$$

and we have used the relations $x\delta(x) = 0$ and $xP\left(\frac{1}{x}\right) = 1$ to simplify the equation (51).

Taking Fourier inverse transform of (51), we get

$$f^{(2)} = -2\frac{\partial f_0}{\partial v_{\parallel}^2} \varphi^{(2)} + 2\frac{\partial^2 f_0}{\partial (v_{\parallel}^2)^2} [\varphi^{(1)}]^2. \tag{53}$$

Substituting for $f^{(2)}$ and $n^{(2)}$ given respectively by (53) and the first equation of (46) into the equation (48), we get the following equation after simplification

$$-\left(v^2 - \frac{5\sigma}{3}\right)^{-1} \left[(1 - \beta) \left(v^2 - \frac{5\sigma}{3}\right) - 1 \right] \varphi^{(2)} + B' [\varphi^{(1)}]^2 = 0. \tag{54}$$

Now the first term of the left hand side of the equation (54) is identically equal to zero due to the dispersion relation as given by the equation (45) and as $B' \approx O(\epsilon^{\frac{1}{2}})$, the second term of the left hand side of the equation (54) along with its sign has to be included in the next higher order Poisson equation, i.e., this term along with its sign must be included in the left hand side Poisson equation of order $\epsilon^{\frac{3}{2}}$ and consequently the equation (54), i.e., the Poisson equation of order ϵ is identically satisfied. So, including the term $+B'[\varphi^{(1)}]^2$ in the Poisson equation of order $\epsilon^{\frac{3}{2}}$, we can write the Poisson equation at the order $\epsilon^{\frac{3}{2}}$ as follows

$$\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \zeta^2} + n^{(3)} - \int_{-\infty}^{\infty} f^{(3)} dv_{\parallel} + B' [\varphi^{(1)}]^2 = 0. \tag{55}$$

Now, in the next order, i.e., at the order $\epsilon^{\frac{1}{2}}$, solving the ion continuity equation and the parallel component of ion fluid equation of motion for the variable $\frac{\partial n^{(3)}}{\partial \zeta}$ to express it in terms $\varphi^{(3)}, \varphi^{(2)}, \varphi^{(1)}$, we get the following equation.

$$\begin{aligned} \frac{\partial n^{(3)}}{\partial \zeta} = & \left(v^2 - \frac{5\sigma}{3}\right)^{-1} \frac{\partial \varphi^{(3)}}{\partial \zeta} + 2V \left(v^2 - \frac{5\sigma}{3}\right)^{-2} \frac{\partial \varphi^{(1)}}{\partial \tau} + \frac{V^4}{\omega_c^2 \left(v^2 - \frac{5\sigma}{3}\right)^2} \frac{\partial}{\partial \zeta} \left[\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} \right] \\ & + \left(v^2 - \frac{5\sigma}{3}\right)^{-4} \left[\frac{10}{27} \sigma - 6V^2 + \frac{3}{2} \frac{(3V^2 - \frac{5\sigma}{9})}{\left(v^2 - \frac{5\sigma}{3}\right)} \right] [\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial \zeta} + \frac{(3V^2 - \frac{5\sigma}{9})}{\left(v^2 - \frac{5\sigma}{3}\right)^3} \frac{\partial}{\partial \zeta} [\varphi^{(1)} \varphi^{(2)}], \end{aligned} \tag{56}$$

where we have used equations (32), (46) and (47) to get this equation in this present form.

Now our task is to find $f^{(3)}$ that determines $n^{(3)}$ from the Poisson equation (55) at the order $\epsilon^{\frac{3}{2}}$. To find $f^{(3)}$ we consider the Vlasov equation at the order ϵ^2 . The Vlasov equation at the order ϵ^2 is the following, in which as in the lowest order case an extra higher order term $\epsilon^4 \alpha_2 \frac{\partial f^{(3)}}{\partial \tau}$ has been included and $f^{(3)}$ has been replaced by $f_{\epsilon}^{(3)}$ and where we have substituted the expressions for $f^{(1)}$ and $f^{(2)}$ given by equations (43) and (53) respectively.

$$\epsilon^2 \alpha_2 \frac{\partial f_{\epsilon}^{(3)}}{\partial \tau} + v_{\parallel} \frac{\partial f_{\epsilon}^{(3)}}{\partial \zeta} + \frac{\partial \varphi^{(3)}}{\partial \zeta} \frac{\partial f_0}{\partial v_{\parallel}} = -2V \alpha_2 \frac{\partial f_0}{\partial v_{\parallel}^2} x_2 + 4v_{\parallel} \frac{\partial^2 f_0}{\partial (v_{\parallel}^2)^2} y_2 - 4v_{\parallel} \frac{\partial^3 f_0}{\partial (v_{\parallel}^2)^3} z_2. \tag{57}$$

where

$$x_2 = \frac{\partial \varphi^{(1)}}{\partial \zeta}, \quad y_2 = \frac{\partial}{\partial \zeta} [\varphi^{(1)} \varphi^{(2)}], \quad z_2 = [\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial \zeta}. \tag{58}$$

Therefore $f^{(3)}$ is obtained from the unique solution of the equation (57) by the relation

$$f^{(3)} = \lim_{\epsilon \rightarrow 0} f_{\epsilon}^{(3)} \tag{59}$$

As in the earlier cases, assuming τ dependence of $f_{\epsilon}^{(3)}$ and $\varphi^{(3)}$ to be of the form $e^{i\omega\tau}$, taking Fourier transform of this equation with respect to the variable ζ , using the causality condition (40) and finally proceeding to the limit $\epsilon \rightarrow 0+$, we get according to (50) the following equation determining $\hat{f}^{(3)}$:

$$\begin{aligned} ik \left[\hat{f}^{(3)} + 2 \frac{\partial f_0}{\partial v_{\parallel}^2} \hat{\varphi}^{(3)} \right] \\ = -2V \alpha_2 \frac{\partial f_0}{\partial v_{\parallel}^2} \left[kP \left(\frac{1}{kv_{\parallel}} + i\pi \operatorname{sgn}(k) \delta(v_{\parallel}) \right) \right] \hat{x}_2 + 4 \frac{\partial^2 f_0}{\partial (v_{\parallel}^2)^2} \hat{y}_2 - 4 \frac{\partial^3 f_0}{\partial (v_{\parallel}^2)^3} \hat{z}_2. \end{aligned} \tag{60}$$

Integrating (60) over the entire range of v_{\parallel} , we get the following equation.

$$ik \left[\hat{n}_e^{(3)} - (1 - \beta) \hat{\varphi}^{(3)} \right] = -\frac{1}{4} iV\alpha_2(4 - 3\beta) \sqrt{\frac{\pi}{2}} \text{sgn}(k) \hat{x}_2 + \hat{y}_2 + \frac{1}{2}(1 + 3\beta) \hat{z}_2, \tag{61}$$

where we set

$$n_e^{(3)} - \int_{-\infty}^{\infty} f^{(3)} dv_{\parallel} = 0. \tag{62}$$

Taking inverse Fourier transform of the above equation, we get

$$\begin{aligned} \frac{\partial n_e^{(3)}}{\partial \zeta} &= (1 - \beta) \frac{\partial \varphi^{(3)}}{\partial \zeta} + \frac{\partial}{\partial \zeta} [\varphi^{(1)} \varphi^{(2)}] + \frac{1}{2} (1 + 3\beta) [\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial \zeta} \\ &\quad - \frac{1}{4\sqrt{2\pi}} V\alpha_2(4 - 3\beta) P \int_{-\infty}^{\infty} \frac{\partial \varphi^{(1)}}{\partial \zeta'} \frac{d\zeta'}{\zeta - \zeta'} \end{aligned} \tag{63}$$

in which the convolution theorem has been used to find the inverse Fourier transform of $\text{sgn}(k) \hat{x}_2$. Now using the equations (62) and (63), we get the following equation.

$$\begin{aligned} \frac{\partial}{\partial \zeta} \left[\int_{-\infty}^{\infty} f^{(3)} dv_{\parallel} \right] &= (1 - \beta) \frac{\partial \varphi^{(3)}}{\partial \zeta} + \frac{\partial}{\partial \zeta} [\varphi^{(1)} \varphi^{(2)}] + \frac{1}{2} (1 + 3\beta) [\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial \zeta} \\ &\quad - \frac{1}{4\sqrt{2\pi}} V\alpha_2(4 - 3\beta) P \int_{-\infty}^{\infty} \frac{\partial \varphi^{(1)}}{\partial \zeta'} \frac{d\zeta'}{\zeta - \zeta'} \end{aligned} \tag{64}$$

Substituting (64) into the equation obtained by differentiating the Poisson equation (55) at the order $\epsilon^{\frac{3}{2}}$ with respect to ζ , we get the following equation

$$\begin{aligned} \frac{\partial n^{(3)}}{\partial \zeta} - (1 - \beta) \frac{\partial \varphi^{(3)}}{\partial \zeta} + 2B' \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial \zeta} - \frac{1}{2} (1 + 3\beta) [\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial \zeta} + \frac{\partial}{\partial \zeta} \left[\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \zeta^2} \right] \\ + \frac{1}{4\sqrt{2\pi}} V\alpha_2(4 - 3\beta) P \int_{-\infty}^{\infty} \frac{\partial \varphi^{(1)}}{\partial \zeta'} \frac{d\zeta'}{\zeta - \zeta'} - \frac{\partial}{\partial \zeta} [\varphi^{(1)} \varphi^{(2)}] = 0 \end{aligned} \tag{65}$$

Now substituting for $\frac{\partial n^{(3)}}{\partial \zeta}$ given by (64) into the equation (65), we get the following further modified macroscopic evolution equation, where the term $+2B' \frac{\partial}{\partial \zeta} [\varphi^{(1)} \varphi^{(2)}]$ being of higher order since $B' = O(\epsilon^{\frac{1}{2}})$ has been omitted.

$$\begin{aligned} \frac{\partial \varphi^{(1)}}{\partial \tau} + AB' \varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial \zeta} + AB'' [\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial \zeta} + \frac{1}{2} A \frac{\partial^3 \varphi^{(1)}}{\partial \zeta^3} + \frac{1}{2} AD \frac{\partial}{\partial \zeta} \left(\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2} \right) \\ + \frac{1}{2} AE\alpha_2 P \int_{-\infty}^{\infty} \frac{\partial \varphi^{(1)}}{\partial \zeta'} \frac{d\zeta'}{\zeta - \zeta'} = 0. \end{aligned} \tag{66}$$

Here A, B', D and B'' are respectively given by the equations (12)-(15) and (22) and the constant V is determined by the equation (16). The equation (66) is a combined MKdV-KdV-ZK equation except for an extra term (last term of the left hand side of (66)) that accounts for the effect of Landau damping. In the next section, we find the solitary wave solution of this further modified macroscopic evolution equation.

4. Solitary wave solution of the further modified macroscopic equation

If we neglect the electron to ion mass ratio, i.e., if we set $\alpha_2 = 0$, the equation (66) reduces to a combined MKdV-KdV-ZK equation. The solitary wave solution of this combined MKdV-KdV-ZK equation has been studied in Das *et al* [2]. In this paper, our aim is to find the solitary wave solution of the equation (66).

The solitary wave solution of the equation (66) with $\alpha_2 = 0$ propagating at an angle δ with the external uniform static magnetic field is the following, which has already been obtained in section IV of Das *et al*. [2],

$$\varphi^{(1)} = \varphi_0(Z) = a \frac{S}{\Psi} \tag{67}$$

where

$$S = \text{sech}[2pZ], \tag{68}$$

$$\Psi = S + \lambda\sqrt{M}, \tag{69}$$

$$\lambda = \pm 1, \tag{70}$$

$$a = \frac{12p^2(\cos^2\delta + D\sin^2\delta)}{12p^2 B'(\cos^2\delta + D\sin^2\delta)}, \tag{71}$$

$$M = 1 + \frac{12p^2 B''(\cos^2\delta + D\sin^2\delta)}{B'^2}, \tag{72}$$

$$Z = \xi \sin\delta + \zeta \cos\delta - U\tau. \tag{73}$$

For the existence of the solitary wave solution (67), it is necessary that the following condition is satisfied.

$$L = MB'^2 = B'^2 + 12p^2B''(\cos^2\delta + D\sin^2\delta) > 0 \tag{74}$$

If the condition (74) holds good, U is given by the equation

$$U = 4p^2a_3 \tag{75}$$

where

$$a_3 = \frac{1}{2}A\cos\delta (\cos^2\delta + D\sin^2\delta) \tag{76}$$

With the help of the equations (71), (72), (75) and (76), we get the following expressions of p, U and M to express them in terms of a :

$$p = \frac{1}{2} \sqrt{\frac{aa_1}{6a_3}} \tag{77}$$

$$U = \frac{1}{6}aa_1 \tag{78}$$

$$M = 1 + a \frac{a_2}{a_1} \tag{79}$$

where

$$a_1 = AB'\cos\delta, \tag{80}$$

$$a_2 = AB''\cos\delta. \tag{81}$$

Using (77), we can write the Eq.(67) as

$$\begin{aligned} \varphi^{(1)} &= \varphi_0(Z) \\ &= a \frac{\operatorname{sech} \left[\sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}aa_1\tau \right) \right]}{\operatorname{sech} \left[\sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}aa_1\tau \right) \right] + \lambda \sqrt{1 + a \frac{a_2}{a_1}}} \end{aligned} \tag{82}$$

Assuming that a to be a slowly varying function of time, following Ott and Sudan [3], we introduced the following space coordinate in a frame moving with the solitary wave.

$$\bar{Z} = \sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}a_1 \int_0^\tau a \, d\tau \right) \tag{83}$$

It is important to note that if a is a constant, then $\bar{Z} = 2pZ$ and consequently,

$$\begin{aligned} \varphi^{(1)} &= \varphi_0(\bar{Z}) \\ &= a \frac{\operatorname{sech} \left[\sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}a_1 \int_0^\tau a \, d\tau \right) \right]}{\operatorname{sech} \left[\sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}a_1 \int_0^\tau a \, d\tau \right) \right] + \lambda \sqrt{1 + a \frac{a_2}{a_1}}} \end{aligned} \tag{84}$$

is the solitary wave solution of the combined MKdV-KdV-ZK equation propagating at an angle δ to the external uniform static magnetic field. Now dropping “overline” on \bar{Z} , we can write the equation (84) as

$$\begin{aligned} \varphi^{(1)} &= \varphi_0(Z) \\ &= a \frac{\operatorname{sech} \left[\sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}a_1 \int_0^\tau a \, d\tau \right) \right]}{\operatorname{sech} \left[\sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}a_1 \int_0^\tau a \, d\tau \right) \right] + \lambda \sqrt{1 + a \frac{a_2}{a_1}}} \end{aligned} \tag{85}$$

where Z is given by the following equation:

$$Z = \sqrt{\frac{aa_1}{6a_3}} \left(\xi \sin\delta + \zeta \cos\delta - \frac{1}{6}a_1 \int_0^\tau a \, d\tau \right) \tag{86}$$

Now our aim is to find the condition for which $\varphi^{(1)}$ given by the equation (85) is a solitary wave solution of the further modified macroscopic equation (66).

With the change of variable defined by the equation (86) and assuming that $\varphi^{(1)}$ is a function of Z, τ only, Eq.(66) can be written as

$$\begin{aligned} \frac{\partial \varphi^{(1)}}{\partial \tau} + \left(-\frac{1}{3}a_1p\alpha + \frac{Z}{2a} \frac{\partial a}{\partial \tau} \right) \frac{\partial \varphi^{(1)}}{\partial Z} + 2pa_1\varphi^{(1)} \frac{\partial \varphi^{(1)}}{\partial Z} + 2pa_2[\varphi^{(1)}]^2 \frac{\partial \varphi^{(1)}}{\partial Z} + 8p^3a_3 \frac{\partial^3 \varphi^{(1)}}{\partial Z^3} \\ + AEa_2p\cos\delta P \int_{-\infty}^{\infty} \frac{\partial \varphi^{(1)}}{\partial Z'} \frac{dZ'}{Z - Z'} = 0 \end{aligned} \tag{87}$$

To investigate the solution of Eq. (87), we follow Ott and Sudan [3] and generalizing the multiple-time scale analysis with respect to α_2 , by setting

$$\varphi^{(1)}(Z, \tau) = q^{(0)} + \alpha_2 q^{(1)} + \alpha_2^2 q^{(2)} + \alpha_2^3 q^{(3)} + \dots \tag{88}$$

where each $q^{(j)}$, ($j = 0, 1, 2, 3, \dots$) are the function of $\tau = \tau_0, \tau_1, \tau_2, \dots$. Here τ_j is given by

$$\tau_j = \alpha_2^j \tau, j = 0, 1, 2, 3, \dots \tag{89}$$

Substituting (88) into (87) and then equating the coefficient of different power of α_2 on each side of Eq. (87), we get a sequence of equations. The zeroth and the first order equation of this sequence are respectively, given by the following equations.

$$\rho \left[\frac{\partial}{\partial \tau} + \frac{Z}{2a} \frac{\partial a}{\partial \tau} \frac{\partial}{\partial Z} \right] q^{(0)} + \bar{L} \frac{\partial}{\partial Z} q^{(0)} = 0 \tag{90}$$

$$\rho \left[\frac{\partial}{\partial \tau} + \frac{Z}{2a} \frac{\partial a}{\partial \tau} \frac{\partial}{\partial Z} \right] q^{(1)} + \bar{L} \frac{\partial}{\partial Z} q^{(1)} = \rho \bar{M} q^{(0)} \tag{91}$$

where

$$\bar{L} = \frac{\partial^2}{\partial Z^2} + \frac{6}{a} q^{(0)} + \frac{6(M-1)}{a^2} [q^{(0)}]^2 - 1 \tag{92}$$

$$\rho = 6 \sqrt{\frac{6a_2}{a_1^3}} a^{-\frac{3}{2}} \tag{93}$$

$$\bar{M} q^{(0)} = - \left[\frac{\partial q^{(0)}}{\partial \tau} + \frac{Z}{2a} \frac{\partial a}{\partial \tau} \frac{\partial q^{(0)}}{\partial Z} + AE p \cos \delta P \int_{-\infty}^{\infty} \frac{\partial q^{(0)}}{\partial Z'} \frac{dZ'}{Z-Z'} \right] \tag{94}$$

Now it can be easily verified that $q^{(0)} = \alpha \frac{\text{sech} Z}{\text{sech} Z + \lambda \sqrt{M}}$ is the soliton solution of the zeroth order equation if

$$\frac{\partial a}{\partial \tau} = 0 \tag{95}$$

which implies that a is independent of time, i.e., at the lowest order, the solitary wave solution of the further modified macroscopic evolution equation is same as that of the combined MKdV-KdV-ZK equation.

Using (95), Eq.(91) can be written as

$$\rho \frac{\partial q^{(1)}}{\partial \tau} + \frac{\partial}{\partial Z} \bar{L} q^{(1)} = \rho \bar{M} q^{(0)} \tag{96}$$

Now for the existence of a solution of the equation (96), its right hand must be perpendicular to the kernel of the operator adjoint to the operator $\frac{\partial}{\partial Z} \bar{L}$; this kernel, which must tend to zero as $|Z| \rightarrow \infty$ is $\frac{\text{sech} Z}{\text{sech} Z + \lambda \sqrt{M}}$. Thus we get the following consistency condition for the existence of a solution of the equation (96).

$$\int_{-\infty}^{\infty} \frac{\text{sech} Z}{\text{sech} Z + \lambda \sqrt{M}} \bar{M} q^{(0)} dZ = 0. \tag{97}$$

From equation (97), we get the following differential equation for the solitary wave amplitude a .

$$\frac{\partial a}{\partial \tau_1} + \frac{AE a^{\frac{3}{2}} (B' + aB'') \cos \delta}{\sqrt{3B'(\cos^2 \delta + D \sin^2 \delta)}} P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(Z) \frac{\partial [\Psi(Z')]}{\partial Z'} \frac{dZ dZ'}{Z-Z'} = 0 \tag{98}$$

where

$$\Psi(Z) = \frac{\text{sech} Z}{\text{sech} Z + \lambda \sqrt{M}} \tag{99}$$

Using the relation $\tau_1 = \alpha_2 \tau$, the equation (ref{5.4.36}) can be written in the following simplified form:

$$\frac{\partial a}{\partial \tau} + \frac{AE \alpha_2 a^{\frac{3}{2}} (B' + aB'') \cos \delta}{\sqrt{3B'(\cos^2 \delta + D \sin^2 \delta)}} P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(Z) \frac{\partial [\Psi(Z')]}{\partial Z'} \frac{dZ dZ'}{Z-Z'} = 0 \tag{100}$$

Here it is important to note that $M \left(= 1 + a \frac{\alpha_2}{\alpha_1} \right)$ appearing in $\Psi(Z)$ is a function of a . So, it is not possible to find the exact analytical dependence of a on τ . But we can solve the above equation by using the Taylor series expansion for the terms of the form $\frac{1}{\text{sech} x + \lambda \sqrt{M}}$ in powers of a . Keeping terms upto the order $a^{\frac{5}{2}}$, we get the following differential equation for a from equation (100).

$$\frac{\partial a}{\partial \tau} + \frac{AE \alpha_2 a^{\frac{3}{2}} \sqrt{B'} \cos \delta}{\sqrt{3(\cos^2 \delta + D \sin^2 \delta)}} \gamma_1 - \frac{AE \alpha_2 a^{\frac{5}{2}} B'' \cos \delta}{\sqrt{3B'(\cos^2 \delta + D \sin^2 \delta)}} (\gamma_2 + \gamma_3) + \frac{AE \alpha_2 a^{\frac{5}{2}} B'' \cos \delta}{\sqrt{3B'(\cos^2 \delta + D \sin^2 \delta)}} \gamma_1 = 0 \tag{101}$$

where $\gamma_1, \gamma_2, \gamma_3$ are given by the following integrals.

$$\gamma_1 = P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_1(Z) \frac{\partial [\Phi_1(Z')]}{\partial Z'} \frac{dZ dZ'}{Z-Z'} = 0, \tag{102a}$$

$$\gamma_2 = P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_2(Z) \frac{\partial [\Phi_2(Z')]}{\partial Z'} \frac{dZ dZ'}{Z-Z'} = 0, \tag{102b}$$

$$\gamma_3 = P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_2(Z) \frac{\partial[\Phi_1(Z')]}{\partial Z'} \frac{dZ dZ'}{Z - Z'} = 0. \tag{102c}$$

$\Phi_1(Z)$ and $\Phi_2(Z)$ appearing in the above are given by

$$\Phi_1(Z) = \frac{\text{sech}Z}{\text{sech}Z + \lambda}, \tag{103}$$

$$\Phi_2(Z) = \frac{\text{sech}Z}{(\text{sech}Z + \lambda)^2}. \tag{104}$$

Now solving the above differential equation (101) for α by the use of the initial condition, $\alpha = \alpha_0$ when $\tau = 0$, we get the following equation for α :

$$\mu \tan^{-1} \left[\frac{\mu(\sqrt{\alpha} - \sqrt{\alpha_0})}{1 + \mu^2 \sqrt{\alpha \alpha_0}} \right] - \frac{\sqrt{\alpha} - \sqrt{\alpha_0}}{\sqrt{\alpha \alpha_0}} = \frac{\tau}{\tau'}, \tag{105}$$

where

$$\tau' = \left[\frac{1}{2} \frac{AE \alpha_2 \sqrt{B'} \cos \delta}{\sqrt{3(\cos^2 \delta + D \sin^2 \delta)} \gamma_1} \right]^{-1} \tag{106}$$

$$\mu = \sqrt{\frac{B''}{B'} \left[1 - \frac{\lambda (\gamma_2 + \gamma_3)}{2 \gamma_1} \right]} \tag{107}$$

From equation (105), we see that α is implicitly depends on τ and consequently, from this equation it is not possible to predict the nature (decreasing or increasing) of dependence of α on τ . But plotting α against τ for the appropriate set of values of the parameters involved in the system, we find that α is slowly varying function of time. By the phrase “appropriate set of values of the parameters”, we mean that those values of the parameters of the system for which the condition for existence of alternative solitary wave solution of the combined MKdV-KdV-ZK equation holds good, i.e., for those values of the parameters of the system for which $L > 0$. Taking $\alpha_0 = 0.5$ (arbitrary) and the values of the parameters as mentioned in the Figure 2, we plot α against τ . This figure clearly shows that the amplitude (α) decays slowly with time (τ) and consequently, the amplitude of the alternative solitary wave solution of the combined MKdV-KdV-ZK equation is a slowly varying function of time when the effect of Landau damping is considered.

5. Conclusions

A macroscopic evolution equation corresponding to the combined MKdV-KdV-ZK equation has been derived to include the effect of Landau damping. This macroscopic evolution equation admits the same alternative solitary wave solution of the combined MKdV-KdV-ZK equation except the fact that the amplitude of the solitary wave solution of the combined MKdV-KdV-ZK like macroscopic equation is a slowly varying function of time. The multiple time scale method of Ott and Sudan [3] has been generalized here to solve the said evolution equation. In small amplitude limit, we have observed the following result.

Result: Due to inclusion of the effect of Landau damping, the amplitude of the alternative solitary wave solution having profile different from sech^2 or sech of the macroscopic evolution equation decays slowly with time.

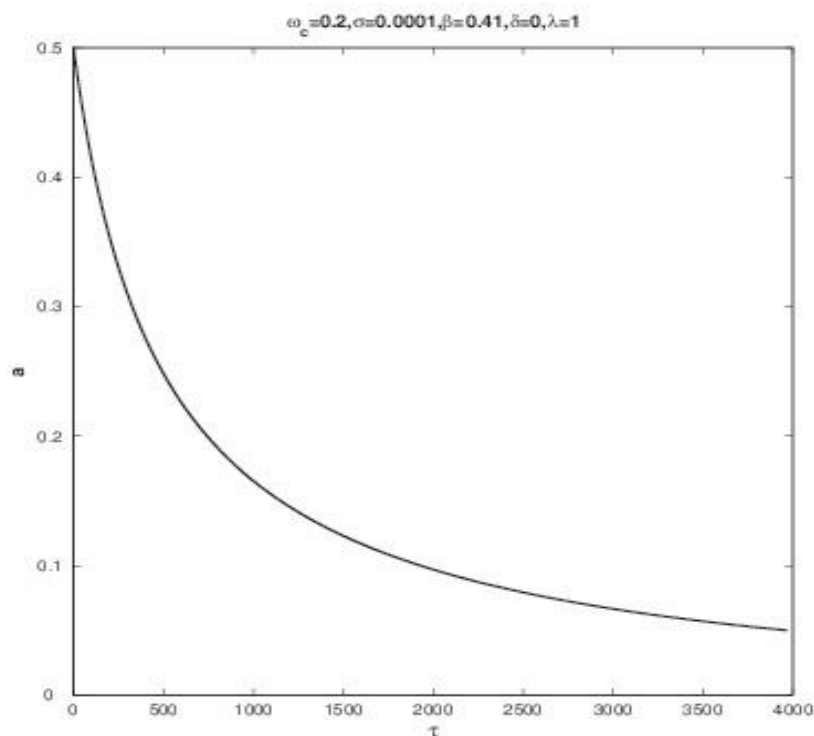


Figure 2: Variation of α against τ

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